Click Comparison of Digital and Matched Filter Receivers

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(Manuscript received May 25, 1967)

We applied the click theory of errors to determine the performance of a digital FM receiver. The receiver had binary orthogonal FSK modulation in a channel that had a single random-phase echo at the symbol duration. We use practical bandwidth assumptions to show that this error performance is identical to that calculated for a matched filter receiver. Numerical results show, for example, that an increase in signal-to-noise ratio of 10 *dB is required to maintain a* 10^{-4} *error rate when an echo of half the signal power is added.*

I. INTRODUCTION

The concept of clicks in an FM receiver was originally used by S. O. Rice[1] and J. Cohn[2] to explain the effect of noise on analog signal demodulation near threshold. Recently, several theoretical investigations of digital FSK signal demodulation have applied the concept of clicks in analyses of low pass filter processing of the discriminator output. Klapper,[3] and Mazo and Salz[4] modelled the low pass filter with an integrate-and-dump function, while Schilling, Hoffman, and Nelson considered a gaussian low pass filter.[5] In all cases, additive gaussian noise was assumed to be the sole source of interference in the signal channel.

In this paper, we consider intersymbol interference that is induced by delay dispersion in the signal channel. Analysis is limited to a practical single-echo channel* and binary orthogonal modulation. Although the analysis seems to be tractable for only special cases, we gain insight into the error mechanism of digital FM reception.

The relation between clicks and errors is viewed as follows. Since the fundamental description of clicks concerns a random angular encircle-

* The single-echo channel was used by Bennett, Curtis, and Rice[6] in their study of analog angle modulated transmission systems.

ment of the origin by the received signal plus noise vector, it is convenient to choose the integrate and dump model for the post-discriminator filter. Then the filter output is a measure of the angular change of the received vector over a symbol duration. Signal and click angular changes are readily compared. The particular signal angular modulation considered here is $\pm\pi$ radians; in this case an error occurs if and only if a click occurs, to a good approximation.† Intersymbol interference is considered as a perturbation of the signal modulation. This distortion affects the instantaneous signal-to-noise ratio and the instantaneous frequency which, as shown by Rice, are the controlling parameters for the click (error) probabilities.

After the calculations described above are used to compute the error rate for the digital FM receiver, another computation for error rate is made using a noncoherent orthogonal matched filter receiver. The error performances of the two receivers are the same for this binary signal having angular modulation $\pm\pi$ radians.

In the following sections, the modulation and the channel parameters are first defined. An expression is derived for the distorted output of the channel. The derivations of the FM receiver performance and the matched filter receiver performance are explained, then the significance of the work is discussed. Two appendices give detailed derivations of the receivers' performances.

II. FSK MODULATION IN THE SINGLE ECHO CHANNEL

Since the two receivers are applied in turn to the same channel as shown in Figure 1, we shall first express the output of this single echo channel. The input waveform is either $s_1(t)$ or $s_2(t)$.

$$\begin{aligned} s_1(t) &= \operatorname{Re}\{e^{i2\pi(f_c+f_d)t}\} \qquad 0 \leqq t \leqq T \\ s_2(t) &= \operatorname{Re}\{e^{i2\pi(f_c-f_d)t}\} \end{aligned} \tag{1}$$

where $s_1(t) = s_2(t) = 0$ otherwise

f_c is a center frequency
f_d is the frequency deviation
T is the symbol duration.

(We shall consider only the deviation: $2f_dT = 1$.) These input waveforms are applied in some arbitrary sequence to the channel; the

† Mazo and Salz[4] have considered the approximations involved in some detail, and their work relates different angular modulations.

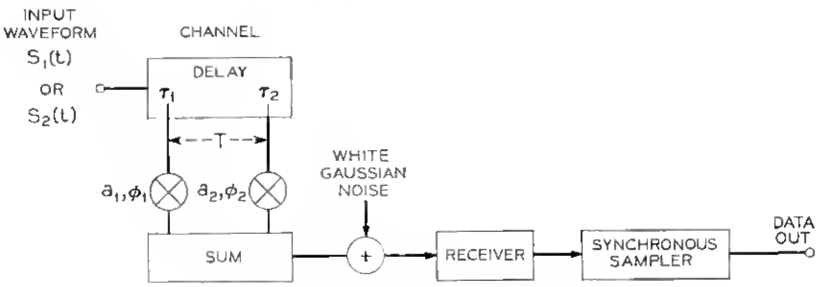

Fig. 1 — Data transmission system model.

channel input voltage can be described as

$$e_{i_n}(t) = \sum_{n=-\infty}^{\infty} s_{i_n}(t - nT) \qquad i_n = 1, 2. \tag{2}$$

The output of the channel (receiver input) with noise added is

$$e(t) = \mathrm{Re}\,\{a_1 e^{i\varphi_1} e_{i_n}(t - \tau_1) + a_2 e^{i\varphi_2} e_{i_n}(t - \tau_2)\} + e_N(t) \tag{3}$$

where

$a_i e^{i\varphi_i}$ $(i = 1, 2)$ are the complex tap gains and we shall consider that $a_1 > a_2$
τ_i are the tap delays and we shall consider the case $\tau_2 - \tau_1 = T$
$e_N(t)$ is additive white Gaussian noise.

In the ensuing work we shall refer to the first term in the braces of equation (3) as the signal, and to the second term as the echo. Since the echo is displaced by one symbol duration, the receiver input in simply a superposition of signal and echo s_i, s_j $(i, j = 1, 2)$. Because each combination is assumed to be equally probable, and because corresponding conditional error probabilities are equal, it is sufficient to evaluate the cases s_1, s_1 and s_1, s_2. Thus the receiver input is either

$$e(t) = \mathrm{Re}\,\{[a_1 e^{i(2\pi f_d t + \varphi_1)} + a_2 e^{i(2\pi f_d t + \varphi_2)}] e^{i2\pi f_c t}\} + e_N(t)$$
$$0 \leqq t \leqq T \tag{4}$$

or

$$e(t) = \{[a_1 e^{i(2\pi f_d t + \varphi_1)} + a_2 e^{i(-2\pi f_d t + \varphi_2)}] e^{i2\pi f_c t}\} + e_N(t). \tag{5}$$

III. FM RECEIVER PERFORMANCE

The FM receiver model used here includes a predetection filter, limiter, discriminator, and a postdetection integrate and dump circuit,

as shown in Figure 2. The predetection filter is used to reduce noise and is supposed to have negligible effect on the modulation. Then the receiver output is proportional to the angular change of the input modulation over a symbol duration.

We proceed by first rewriting the input voltage, represented by equation (4) or equation (5), in a form that shows explicitly the amplitude and angular variations of the noise-free envelope in the form

$$e(t) = \mathrm{Re}\ \{A(t)e^{i\varphi(t)}e^{i2\pi f_c t}\}.$$

The envelope which describes the signal-echo pair s_1, s_1, corresponding to equation (4), is

$$\begin{aligned} A &= [a_1^2 + a_2^2 + 2a_1a_2 \cos(\varphi_1 - \varphi_2)]^{\frac{1}{2}} \\ \varphi(t) &= 2\pi f_d t + \varphi_0 \end{aligned} \tag{6}$$

where φ_0 is a constant. The envelope which describes the signal-echo pair s_1, s_2, corresponding to equation (5), is

$$\begin{aligned} A(t) &= [a_1^2 + a_2^2 + 2a_1a_2 \cos(4\pi f_d t + \varphi_1 - \varphi_2)]^{\frac{1}{2}} \\ \varphi(t) &= 2\pi f_d t + \varphi_1 - \tan^{-1}\left[\frac{a_2 \sin(4\pi f_d t + \varphi_1 - \varphi_2)}{a_1 + a_2 \cos(4\pi f_d t + \varphi_1 - \varphi_2)}\right]. \end{aligned} \tag{7}$$

(In these equations, $A(t)$ and $\varphi(t)$ have been obtained by straightforward trigonometric relations from equations (4) and (5).)

Thus in the absence of noise for signal and echo pairs s_1, s_1 or s_1, s_2, the receiver output is proportional to

$$\Delta\varphi = \varphi(T) - \varphi(0) = 2\pi f_d T = +\pi \qquad (a_2 < a_1),$$

Similarly, complementary signal-echo pair s_2, s_2 or s_2, s_1 would give an output $\Delta\varphi = -\pi$.

The noise perturbation is considered an additive error angle $\theta(t)$, illustrated in Fig. 3. Now the FM receiver output is proportional to $\psi(t)$

$$\psi(t) = \varphi(t) + \theta(t)$$

Fig. 2 — Digital FM receiver.

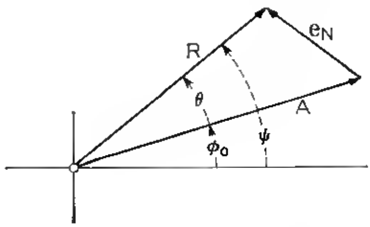

Fig. 3 — Angular perturbation caused by noise.

and the angular change over a symbol duration is

$$\Delta\psi = \Delta\varphi + \Delta\theta.$$

The decision threshold is placed at $\Delta\psi = 0$, midway between $\pm\pi$. When the transmitted signal has an angular modulation $\Delta\varphi = +\pi$, an error is made if $\Delta\theta < -\pi$.

Fig. 4 illustrates possible loci of the signal plus noise envelope R. For signal alone, the locus is simply a semicircle. With echo and noise added, no error is made provided $\Delta\psi > 0$. We observe that the locus encircles the origin in a counterclockwise direction. But when a negative click occurs, the locus encircles the origin in a clockwise direction, $\Delta\psi < 0$, and an error is made.

The probability of error is obtained from the probability of a negative click during a symbol interval. Rice[4] defines $H_- \, dt$ as the proba-

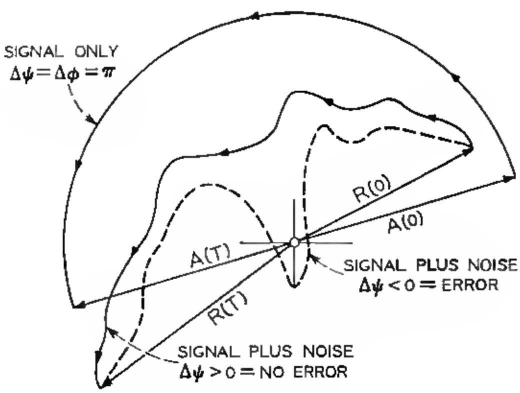

Fig. 4 — Possible loci of angular change.

bility of a negative click: the angle $\theta(t)$ decreases* through an odd multiple of π between t and $t + dt$. $H_- \, dt$ is a function of signal-to-noise ratio and the time derivative of the angular modulation, which are time- and phase-dependent according to equations (6) or (7). The desired probability of error is obtained by integrating H_- over a symbol duration and averaging over the random channel phase angle:

$$P = \frac{1}{2\pi} \int_0^{2\pi} dx \int_0^T H_-(\rho, \dot{\varphi}) \, dt \tag{8}$$

where

$\rho = \dfrac{A^2(t)}{2\overline{e_N^2}}$ is the instantaneous signal-to-noise ratio

$\overline{e_N^2}$ is the noise power passed by tbe predetection filter

$\dot{\varphi}$ is the time derivative of tbe modulation angle $\varphi(t)$

$x = \varphi_1 - \varphi_2$ is tbe relative echo pbase, assumed uniformly distributed over $(0, 2\pi)$.

As Appendix A shows, the error probability obtained when $A(t)$ and $\varphi(t)$ from equation (6) are substituted in equation (8) is

$$P_{e1} = \frac{1}{2} I_0\left(\frac{a_1 a_2}{\overline{e_N^2}}\right) \exp\left(-\frac{a_1^2 + a_2^2}{2\overline{e_N^2}}\right). \tag{9}$$

The error probability corresponding to equation (7) is

$$P_{e2} = Q\left[\frac{a_2}{(\overline{e_N^2})^{\frac{1}{2}}}, \frac{a_1}{(\overline{e_N^2})^{\frac{1}{2}}}\right] - \frac{1}{2} I_0\left(\frac{a_1 a_2}{\overline{e_N^2}}\right) \exp\left(-\frac{a_1^2 + a_2^2}{2\overline{e_N^2}}\right), \tag{10}$$

where $Q[\cdot,\cdot]$ is the Marcum Q function. The average of P_{e1} and P_{e2} is simply

$$P_e = \tfrac{1}{2}(P_{e1} + P_{e2}) = \frac{1}{2} Q\left[\frac{a_2}{(\overline{e_N^2})^{\frac{1}{2}}}, \frac{a_1}{(\overline{e_N^2})^{\frac{1}{2}}}\right]. \tag{11}$$

Tbe noise power $\overline{e_N^2}$ depends on the predetection filter bandwidth, which can be estimated using Carson's rule with the Nyquist criterion for video bandwidth. These assumptions give a bandwidth B

$$B = \frac{1}{T}(1 + 2f_d T) = \frac{2}{T} \, Hz \tag{12}$$

*Decrease means in a direction opposite the time derivative of the modulation $\varphi(t)$. It is possible that $\theta(t)$ can also increase by π and thus cancel the decrease; the probability of this occurrence is asymptotically negligible for low error rates.

and thus a noise power

$$\overline{e_N^2} = BN \text{ watts} \tag{13}$$

where N is the noise density in watts/Hz. Substitution of equations (12) and (13) into equation (11) gives

$$P_e = \frac{1}{2} Q\left[a\left(\frac{E}{N}\right)^{\frac{1}{2}}, \left(\frac{E}{N}\right)^{\frac{1}{2}}\right] \tag{14}$$

where

$a = \dfrac{a_2}{a_1}$ is the echo/signal voltage ratio

$E = \frac{1}{2}a_1^2 T$ is the signal energy/bit.

IV. MATCHED FILTER RECEIVER PERFORMANCE

We are concerned here with the incoherent matched filter receiver shown in Fig. 5. The mark and space filters are matched (except for phase) to the waveforms $s_1(t)$ and $s_2(t)$ defined by equation (1). As Fig. 5 indicates, the combined operations of filtering, square law rectifying, and time sampling produce R_1^2 and R_2^2 which are the squared envelopes of the filter outputs at the end of the symbol interval. Assuming mark is transmitted, the probability of error is

$$P = \text{Prob}\,\{R_2^2 > R_1^2\} = \int_0^\infty dR_1 \int_{R_1}^\infty p(R_1\,, R_2)\, dR_2 \tag{15}$$

where $p(R_1, R_2)$ is the joint density function of R_1 and R_2.

As shown in Appendix B, the error probability corresponding to equation (4), with the signal-echo pair s_1, s_1 is

$$P_{e1} = \frac{1}{2} I_0\left(\frac{aE}{N}\right) \exp\left(-\frac{E + a^2E}{2N}\right). \tag{16}$$

The error probability corresponding to equation (5), with the signal-echo pair s_1, s_2 is

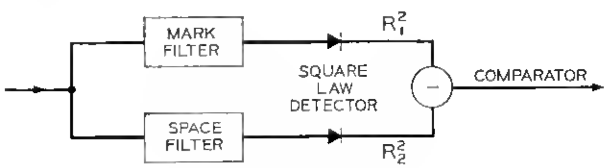

Fig. 5 — Matched filter receiver.

$$P_{e2} = Q\left[a\left(\frac{E}{N}\right)^{\frac{1}{2}}, \left(\frac{E}{N}\right)^{\frac{1}{2}}\right] - \frac{1}{2} I_0\left(\frac{aE}{N}\right) \exp\left(-\frac{E + a^2E}{2N}\right). \tag{17}$$

The average of P_{e1} and P_{e2} is

$$P_e = \tfrac{1}{2}(P_{e1} + P_{e2}) = \frac{1}{2} Q\left[a\left(\frac{E}{N}\right)^{\frac{1}{2}}, \left(\frac{E}{N}\right)^{\frac{1}{2}}\right]. \tag{18}$$

This is identical with the error performance of the FM receiver, specified in equation (14).

V. DISCUSSION OF RESULTS

The concept of clicks has made possible a unique comparison between digital FM and matched filter receivers. When a suitable predetection filter is chosen for the FM receiver and the assumption made that this filter does not significantly process the signal, then the error performance of the two receivers is described identically.

We have gained particular insight into the error mechanism of the digital FM receiver under conditions of intersymbol interference. The analysis shows how the rate of occurrence of the noise clicks is critically dependent on this distortion of the signal waveform. This is in direct contrast to the usual AM systems where intersymbol interference manifests itself by a gradual degradation caused by "eye" closing.

Numerical results, illustrated in Fig. 6, show that the receivers'

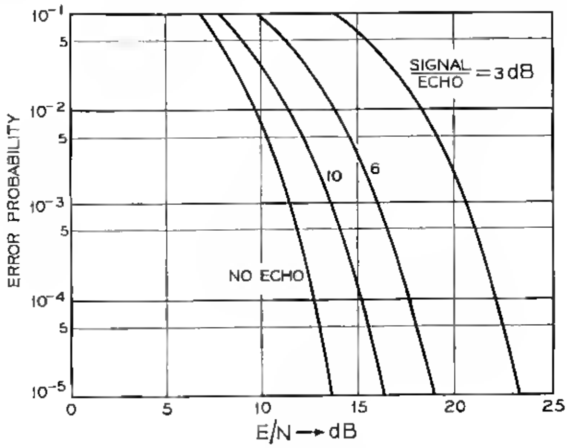

Fig. 6 — Error performance in single echo channel.

performance can be summarized in terms of the increase in signal-to-noise ratio required to maintain a desired error rate when an echo is superposed. For example, 10 dB higher signal-to-noise ratio is required to maintain a 10^{-4} error rate when an echo having half the signal power is added. It is easy to show that the asymptotic deterioration in performance with echo behaves as 20 $\log_{10}$ $(1-a)$dB.

VI. ACKNOWLEDGMENT

The author wishes to thank J. E. Mazo and J. Salz for discussions of this work.

APPENDIX A

Click Probabilities

This appendix concerns the application of an average click prohablity to FM receiver performance. The mathematical details are given here which relate equations (6), (7), and (8) to equations (9) and (10).

It has been shown in Refs. 1 and 2 that the probahility of a click in time dt can be approximated as*

$$H_{-}\, dt \cong \frac{\dot{\varphi}}{2\pi} e^{-\rho}\, dt \tag{19}$$

where $\dot{\varphi}$ is the time derivative of the envelope angular variation and

$$\rho = \frac{A^2}{2\overline{e_N^2}}$$

is the signal-to-noise ratio. Substitution in equation (8) gives the error probability

$$P = \int_0^{2\pi} \frac{dx}{2\pi} \int_0^{T} \frac{dt}{2\pi}\, \dot{\varphi}(x,\, t) \exp\left[-\frac{A^2(x,\, t)}{2\overline{e_N^2}}\right] \tag{20}$$

where $x = \varphi_1 - \varphi_2$. Substitution in equation (20) of $A(x, t)$ and the time derivative of $\varphi(x, t)$, $\dot{\varphi}(x, t) = 2\pi f_d$, from equation (6) gives for

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* This validity of this approximation depends on sufficiently large ρ for $\dot{\varphi} \neq 0$. Klapper[3] has discussed this in some detail. Although the approximation is not as good with intersymbol interference, it appears adequate.

the signal-echo pair s_1, s_1

$$P_{e1} = \frac{1}{2\pi}\int_0^{2\pi} dx \int_0^T f_d \exp\left[-\frac{1}{2\overline{e_N^2}}(a_1^2 + a_2^2 + 2a_1a_2\cos x)\right] dt$$
$$= \frac{1}{2} I_0\left(\frac{a_1a_2}{\overline{e_N^2}}\right)\exp\left(-\frac{a_1^2 + a_2^2}{2\overline{e_N^2}}\right) \tag{21}$$

where we have used the integral definition of $I_0(\cdot)$ and noted $f_dT = \frac{1}{2}$.

We now take $A(x, t)$ and $\varphi(x, t)$ from equation (7) and notice that the time derivative of the latter can be expressed as

$$\dot{\varphi}(t) = 4\pi f_d\left[\frac{a_1^2 + a_1a_2\cos(4\pi f_dt + \varphi_1 - \varphi_2)}{a_1^2 + a_2^2 + 2a_1a_2\cos(4\pi f_dt + \varphi_1 - \varphi_2)}\right] - 2\pi f_d . \tag{22}$$

When the first term of equation (22) and the expression for $A(x, t)$ are substituted into equation (20) the integral can be recognized as a Q function representation given by Helstrom:[7]

$$Q(\alpha, \beta) = \frac{1}{2\pi}\exp\left(-\frac{\alpha^2 + \beta^2}{2}\right)\int_0^{2\pi}\frac{\left(1 - \frac{\alpha}{\beta}\cos u\right)e^{\alpha\beta\cos u}}{1 + \left(\frac{\alpha}{\beta}\right)^2 - 2\left(\frac{\alpha}{\beta}\right)\cos u}\, du$$

where $\alpha < \beta$ and we have replaced $4\pi f_dt + \varphi_1 - \phi_2 + \pi$ by the variable u.

We make the identifications

$$\alpha \triangleq \frac{a_2}{(\overline{e_N^2})^{\frac{1}{2}}}, \qquad \beta \triangleq \frac{a_1}{(\overline{e_N^2})^{\frac{1}{2}}}.$$

Substitution of the second term of equation (22) gives an integral identical to the right-hand side of equation (21). Thus, for the signal-echo pair s_1, s_2

$$P_{e2} = Q\left[\frac{a_2}{(\overline{e_N^2})^{\frac{1}{2}}}, \frac{a_1}{(\overline{e_N^2})^{\frac{1}{2}}}\right] - \frac{1}{2} I_0\left(\frac{a_1a_2}{\overline{e_N^2}}\right)\exp\left(-\frac{a_1^2 + a_2^2}{2\overline{e_N^2}}\right).$$

APPENDIX B

Matched Filter Receiver

This appendix concerns the application of the single echo channel waveforms to filters which are matched to the waveforms in the absence of the echo. The mathematical form is similar to the form

illustrated by Helstrom,[7] and we give only a brief summary here to show the effect of the echo.

The error probability depends on a comparison of the sampled outputs, R_1 and R_2, of the two matched filters. Helstrom shows

$$R_i^2 = X_i^2 + Y_i^2 , \qquad i = 1, 2 \tag{23}$$

where

$$X_i = \int_0^T e(t) \cos (2\pi f_i t) \, dt \tag{24}$$

$$Y_i = \int_0^T e(t) \sin (2\pi f_i t) \, dt \tag{25}$$

where $e(t)$ is signal plus echo plus noise defined by equation (4) or (5).

f_i is one of the signal frequencies
$f_1 = f_c + f_d$
$f_2 = f_c - f_d$.

Substitution of $e(t)$ as given in equation (4) for the signal-echo pair s_1, s_1 gives

$$\begin{aligned} X_1 &= \frac{a_1 T}{2} \cos \varphi_1 + \frac{a_2 T}{2} \cos \varphi_2 + I_{N1} \\ Y_1 &= -\frac{a_1 T}{2} \sin \varphi_1 - \frac{a_2 T}{2} \sin \varphi_2 + I_{N2} , \\ X_2 &= I_{N3} \\ Y_2 &= I_{N4} \end{aligned} \tag{26}$$

where $I_{N1}, I_{N2}, I_{N3}, I_{N4}$ are zero mean independent Gaussian variables having equal variances $\sigma^2 = NT/4$ for noise density N(watts/Hz).

From these terms, we find that the joint distribution of R_1 and R_2 is $p(R_1, R_2)$

$$p(R_1 , R_2) = \frac{R_1 R_2}{\sigma^4} I_0\left(\frac{R_1 C_1}{\sigma^2}\right) \exp\left(-\frac{R_1^2 + R_2^2 + C_1^2}{2\sigma^2}\right) \tag{27}$$

where

$$C_1^2 = \left(\frac{a_1 T}{2}\right)^2 + \left(\frac{a_2 T}{2}\right)^2 + 2\left(\frac{a_1 T}{2}\right)\left(\frac{a_2 T}{2}\right) \cos (\varphi_1 - \varphi_2).$$

The error probability is

$$\begin{aligned} P &= \int_0^\infty dR_1 \int_{R_1}^\infty dR_2\, p(R_1, R_2) \\ &= \frac{1}{2} \exp\left(-\frac{C_1^2}{4\sigma^2}\right). \end{aligned} \tag{28}$$

Averaging this value over $x = \varphi_1 - \varphi_2$ yields

$$P_{e1} = \frac{1}{2\pi} \int_0^{2\pi} P(x)\, dx = \frac{1}{2} I_0\left(\frac{aE}{N}\right) \exp\left[-\frac{E + a^2E}{2N}\right]. \tag{29}$$

Similarly, substitution of $e(t)$ as given by equation (5) into equations (24) and (25) gives, for the signal-echo pair s_1, s_2

$$\begin{aligned} X_1 &= \frac{a_1T}{2} \cos \varphi_1 + I_{N1} \\ Y_1 &= -\frac{a_1T}{2} \sin \varphi_1 + I_{N2} \\ X_2 &= \frac{a_2T}{2} \cos \varphi_2 + I_{N3} \\ Y_2 &= -\frac{a_2T}{2} \sin \varphi_2 + I_{N4}\,. \end{aligned} \tag{30}$$

Now the joint distribution of R_1 and R_2 is found to be

$$p(R_1, R_2) = \frac{R_1R_2}{\sigma^4} I_0\left(\frac{R_1C_1}{\sigma^2}\right) I_0\left(\frac{R_2C_2}{\sigma^2}\right) \exp\left(-\frac{R_1^2 + R_2^2 + C_1^2 + C_2^2}{2\sigma^2}\right) \tag{31}$$

where

$$\begin{aligned} C_1 &= \frac{a_1T}{2} \\ C_2 &= \frac{a_2T}{2}. \end{aligned}$$

The error probability in this case is found via the following steps.

$$\begin{aligned} P_{e2} &= \int_0^\infty dR_1 \int_{R_1}^\infty dR_2\, p(R_1, R_2) \\ &= \int_0^\infty dR_1 \frac{R_1}{\sigma^2} I_0\left(\frac{R_1C_1}{\sigma^2}\right) Q\left(\frac{C_2}{\sigma}, \frac{R_1}{\sigma}\right) \exp\left(-\frac{R_1^2 + C_1^2}{2\sigma^2}\right) \end{aligned} \tag{32}$$

where we have substituted $p(R_1, R_2)$ from equation (31) and used the Q function definition:

$$Q(\alpha, \beta) = \int_\beta^\infty tI_0(\alpha t) \exp\left(-\frac{\alpha^2 + t^2}{2}\right) dt.$$

We see, by this manipulation, that equation (32) is integrable; for example, as shown by Stein.[8] Thus

$$P_{e2} = Q\left(\frac{C_2}{\sigma\sqrt{2}}, \frac{C_1}{\sigma\sqrt{2}}\right) - \frac{1}{2} I_0\left(\frac{C_1C_2}{2\sigma^2}\right) \exp\left(-\frac{C_1^2 + C_2^2}{4\sigma^2}\right). \quad (33)$$

Appropriate substitutions of the terms from equations (31) and (14) give equation (17).

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